Elementary Geometry
Session 1

Topic	Activity Name	Page Number	Related SOL	Activity Sheets	Materials
van Hiele Theory of Geometric Thought	Lecture – van Hiele Levels of Geometric Thought Triangle Sorts	3	K.11, K.12, K.13, 1.16 3.18, 4.17, 5.15	Explanation Sheets -van Hiele levels, -Additional Points, Triangle Sorting Pieces, Pages 1-4 Sample Student Sorts 1-6	Paper triangles
Quadrilaterals and Their Properties	Quadrilateral Sort	19	K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15	Quadrilateral Sorting Pieces	Paper quadrilaterals
	What's My Rule?	22	K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15	What's My Rule?	Paper quadrilaterals
	Quadrilateral Properties Laboratory	24	K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15	Types of Quadrilaterals	Geo-strips, D-stix, or miniature marshmallows and toothpicks; square corner
	Quadrilateral Sorting Laboratory	28	K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15	Quadrilateral Sorting Laboratory, Quadrilateral Table, Quadrilateral Family Tree	Paper quadrilaterals

Topic: The van Hiele Theory of Geometric Thought

Description: The van Hiele theory of geometric thought describes how students learn geometry and provides a framework for structuring student experiences that should lead to conceptual growth and understanding. In this first session, the participants will explore the van Hiele levels of geometric thought by doing triangle sorts and comparing their sorts to those performed by elementary students. The sorting task is appropriate for all ages and levels of students. It can serve as an activity to help students advance their level of understanding as well as an assessment tool that can inform the teacher at what van Hiele level the student is thinking.

Related SOL: K.11, K.12, K.13, 1.16, 3.18, 4.17, 5.15

Activity: The van Hiele Levels of Geometric Thought

Format: Large Group Lecture and Small Group Activity

Objectives: Participants will be able to describe the developmental sequence of geometric thinking according to the van Hiele theory of geometric thought and activities suitable for each level. In addition, participants will be able to assess the van Hiele levels of their students.

Related SOL: K.11, K.12, K.13, 1.16, 3.18, 4.17, 5.15

Materials: Paper triangles, cut out and placed in a plastic baggy or manila envelope (see Activity Sheet for Triangle Sorting Pieces.) You will need at least one set of triangles for every three participants.

Time Required: Approximately 1 hour

Background: To Trainer (for lecture):
After observing their own students, Dutch teachers P.M. van Hiele and Dina van Hiele-Geldof described learning as a discontinuous process with jumps that suggest "levels." They identified five sequential levels of geometric understanding or thought:

1) Visualization
2) Analysis
3) Abstraction
4) Deduction
5) Rigor

Clements and Battista (1992) proposed the existence of a Level 0 that they called Pre-recognition.

In Kindergarten through grade two most students will be at Level 1. By grade three, students should be transitioning to Level 2. If the content in the Virginia Standards of Learning is mastered, students should attain Level 3 by the end of sixth grade. Level 4 is usually attained by students who can prove theorems using deductive techniques. One problem is that most current textbooks provide activities requiring only Level 1 thinking up through sixth grade and teachers must provide different types of tasks to facilitate the development of the higher levels of thought.

Directions:

I. Participants should study the The van Hiele Levels Explanation Sheet.

2. Turn to the Additional Points Explanation Sheet. Note for Point 1 that the levels are hierarchical. Students cannot be expected to write a geometric proof successfully unless they have progressed through each level of thought in turn. At Point 2, college students and even some teachers have been found who are at Level 1, while there are middle schoolers at Level 3 and above. (If the content in the SOL is mastered, students should attain Level 3 by the end of 6th grade.) As an example of an experience that can impede progress (Point 3), think of the illustration of the teacher who knew that the relationship between squares and rectangles was a difficult one for her fourth graders so she had them memorize, "Every square is a rectangle, but not every rectangle is a square." When tested a few weeks later, half the students remembered that a square is a type of rectangle, while the other half thought that a rectangle was a type of square. It was almost impossible for these students to learn the true relationship between squares and rectangles because every time they heard the words square and rectangle together, they insisted on relying on their memorized sentence rather than on the properties of the two types of figures.

3. Continue on to Properties of Levels. As an example of separation, consider the meaning of the word "square." When a teacher thinking at Level 3 or above says "square", the word conveys the properties and relationships of a square: having four congruent sides; having four congruent angles; having perpendicular diagonals; and being a type of polygon, quadrilateral, parallelogram, and rectangle. To a student thinking at Level 1, the word "square" will only evoke an image of something that looks like a square such as a CD case or first base. The same word is being used, but it has an entirely different meaning to the teacher and the student. The teacher must keep in mind what the meaning of the word or symbol is to the student and how the student thinks about it. For Attainment, it is important to note that there are five phases of learning that lead to understanding at the next higher level.

4. Divide participants into small groups. Distribute the sets of cutout triangles, at least one set per three participants. Instruct the participants to lay out the pieces with the letters up. Do not call them triangles. Tell the participants that the objects can be grouped together in many different ways. For example, if we sorted the figures that make up the American flag (the red stripes, the white stripes, the blue field, the white stars), we might sort by color and put the white stripes and the stars together because they

are white, the red stripes in another group because they are red, and the blue field by itself because it is the only blue object. Another way to sort the flag parts would be to put all the stripes and the blue field together because they are all rectangles and all the stars together because they are not rectangles. If needed, you can demonstrate a triangle sort using pieces cut from the Triangle Sorting Pieces Activity Sheet. Have participants sort the figures into groups that belong together, recording the letters of the pieces they put together and the criteria they used to sort. Have them sort two or three times, recording each sort.

5. Ask the participants to describe their sorts. Expect answers like "acute, right, and obtuse triangles" or "scalene, isosceles, and equilateral". Have them compare their sorts with those of other groups.

6. Ask them how they think their students would sort these figures. Refer to Sample Student Sort Sheets and ask the participants to conjecture the criteria used for sorting and the van Hiele level of the sorter. Sample Student Sort 1 is a low Level 1 sort where the student is sorting strictly by size and may not even know that the figures are triangles. Sample Student Sort 2 is another Level 1 sort. Here the student thinks that triangles must have at least two sides the same length or possibly that triangles must be symmetric. Sample Student Sort 3 is another Level 1 sort. This student also believes that triangles must have at least two sides the same length or possibly that triangles must be symmetric. Additionally, this student recognized the figures with right angles or "corners" as a separate category. The Sample Student Sort 4 is at least a Level 2 or 3 sort in which the sorter focuses on the lengths of the sides, a criterion that separates the figures into categories that overlap. The student has actually sorted into groups with no sides the same length, two sides the same length, and all sides the same length. It is unclear whether the student knows that equilateral triangles are a type of isosceles triangle. The Sample Student Sort 5 focuses on parts of the figures and so is a Level 2 sort, but the student does not have the vocabulary to adequately describe the figures. The Sample Student Sort 6 is similar to the Sample Student Sort 4, but the word "Perfect" is incorrect and indicates that the student may be thinking more of the figure as a whole rather than of the individual parts. This sort is probably Level 2.

Explanation Sheet: The van Hiele Levels

Level 1: Visualization. Geometric figures are recognized as entities, without any awareness of parts of figures or relationships between components of the figure. A student should recognize and name figures, and distinguish a given figure from others that look somewhat the same. “I know it's a rectangle because it looks like a door and I know that the door is a rectangle."

Level 2: Analysis. Properties are perceived, but are isolated and unrelated. A student should recognize and name properties of geometric figures. “I know it's a rectangle because it is closed, it has four sides and four right angles, opposite sides are parallel, opposite sides are congruent, diagonals bisect each other, adjacent sides are perpendicular,...”

Level 3: Abstraction. Definitions are meaningful, with relationships being perceived between properties and between figures. Logical implications and class inclusions are understood, but the role and significance of deduction is not understood. “I know it’s a rectangle because it's a parallelogram with right angles."

Level 4: Deduction. The student can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. A student should be able to supply reasons for steps in a proof.

Level 5: Rigor. The standards of rigor and abstraction represented by modern geometries characterize level 5. Symbols without referents can be manipulated according to the laws of formal logic. A student should understand the role and necessity of indirect proof and proof by contrapositive.

Explanation Sheet: **Additional Points**

1. The learner cannot achieve one level without passing through the previous levels.

2. Progress from one level to another is more dependent on educational experience than on age or maturation.

3. Certain types of experiences can facilitate or impede progress within a level or to a higher level.

Properties of Levels

Adjacency: What was intrinsic in the preceding level is extrinsic in the current level.

Distinction: Each level has its own linguistic symbols and its own network of relationships connecting those symbols.

Separation: Two individuals reasoning at different levels cannot understand one another.

Attainment: The learning process leading to complete understanding at the next higher level has five phases: inquiry/information, directed orientation, explication, free orientation and integration.

Phases of Learning

Inquiry/Information: Gets acquainted with the working domain (e.g., examines examples and non-examples)

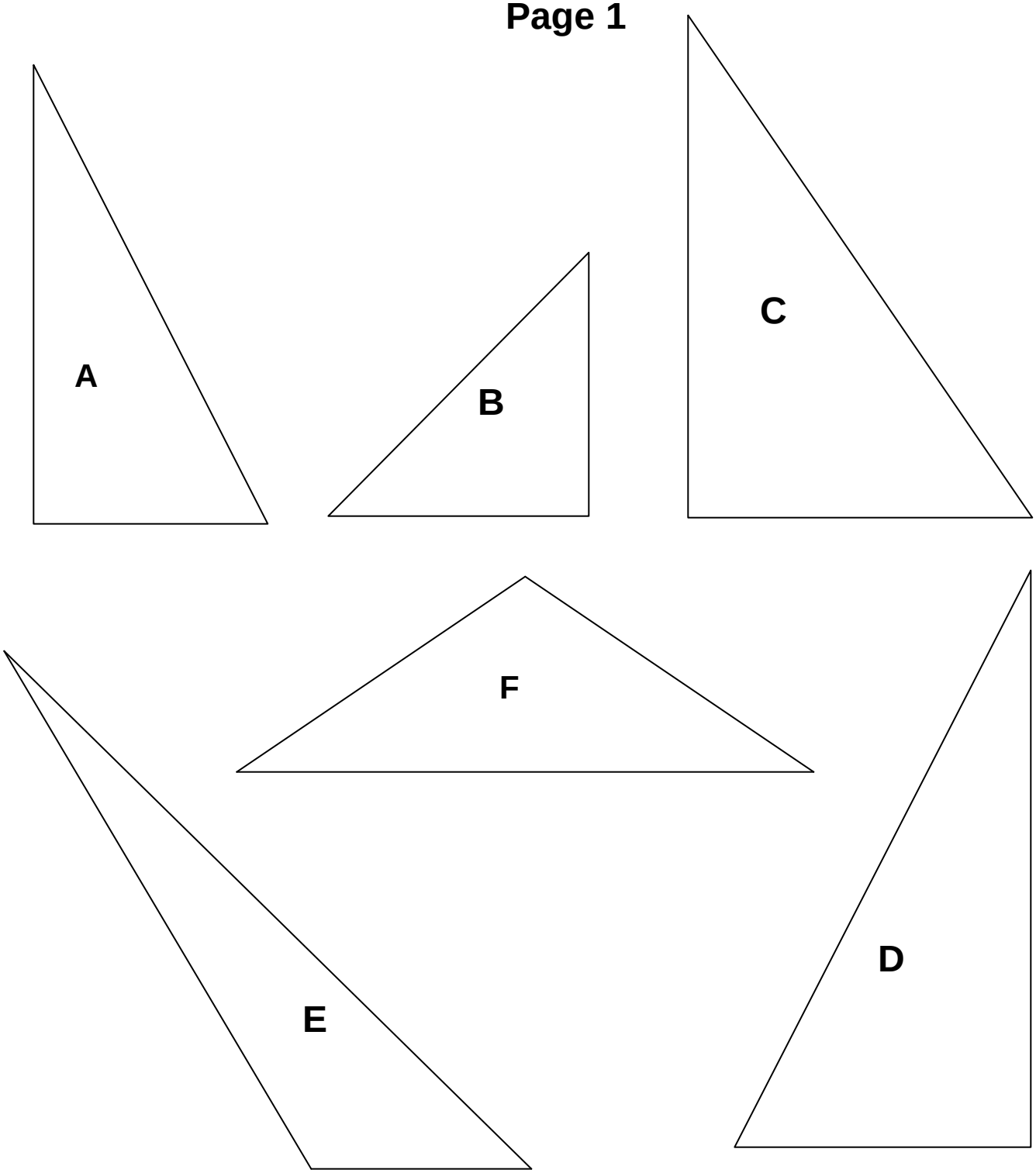
Guided orientation: Does tasks involving different relations of the network that is to be formed (e.g., folding, measuring, looking for symmetry)

Explication: Becomes conscious of the relations, tries to express them in words, and learns technical language which accompanies the subject matter (e.g., expresses ideas about properties of figures)

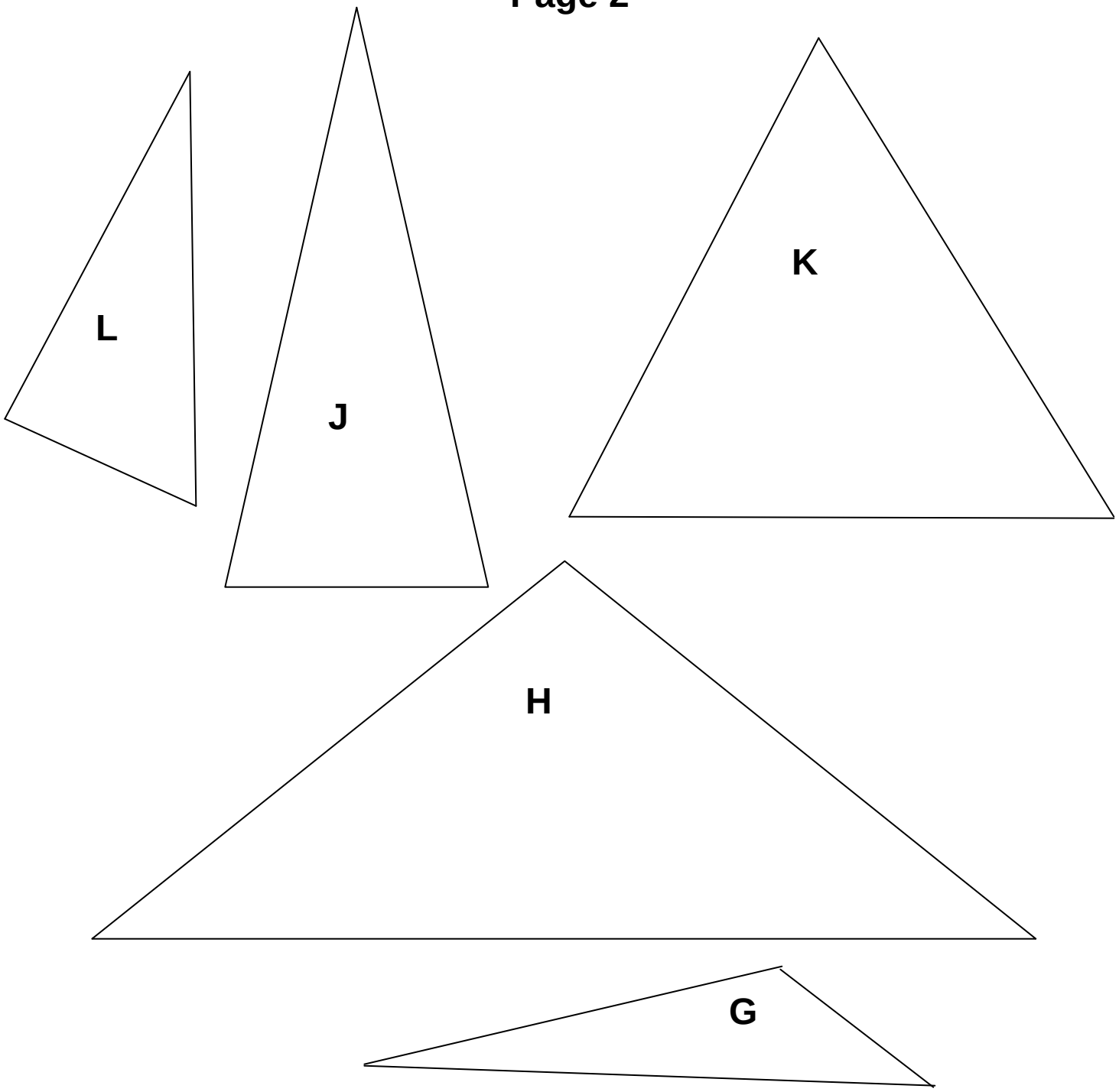
Free orientation: Learns, by doing more complex tasks, to find his/her own way in the network of relations (e.g., knowing properties of one kind of figure, investigating these properties for a new figure, such as kites)

Integration: Summarizes all that has been learned about the subject, then reflects on actions, and obtains an overview of the newly formed network of relations now available (e.g., properties of a figure are summarized)

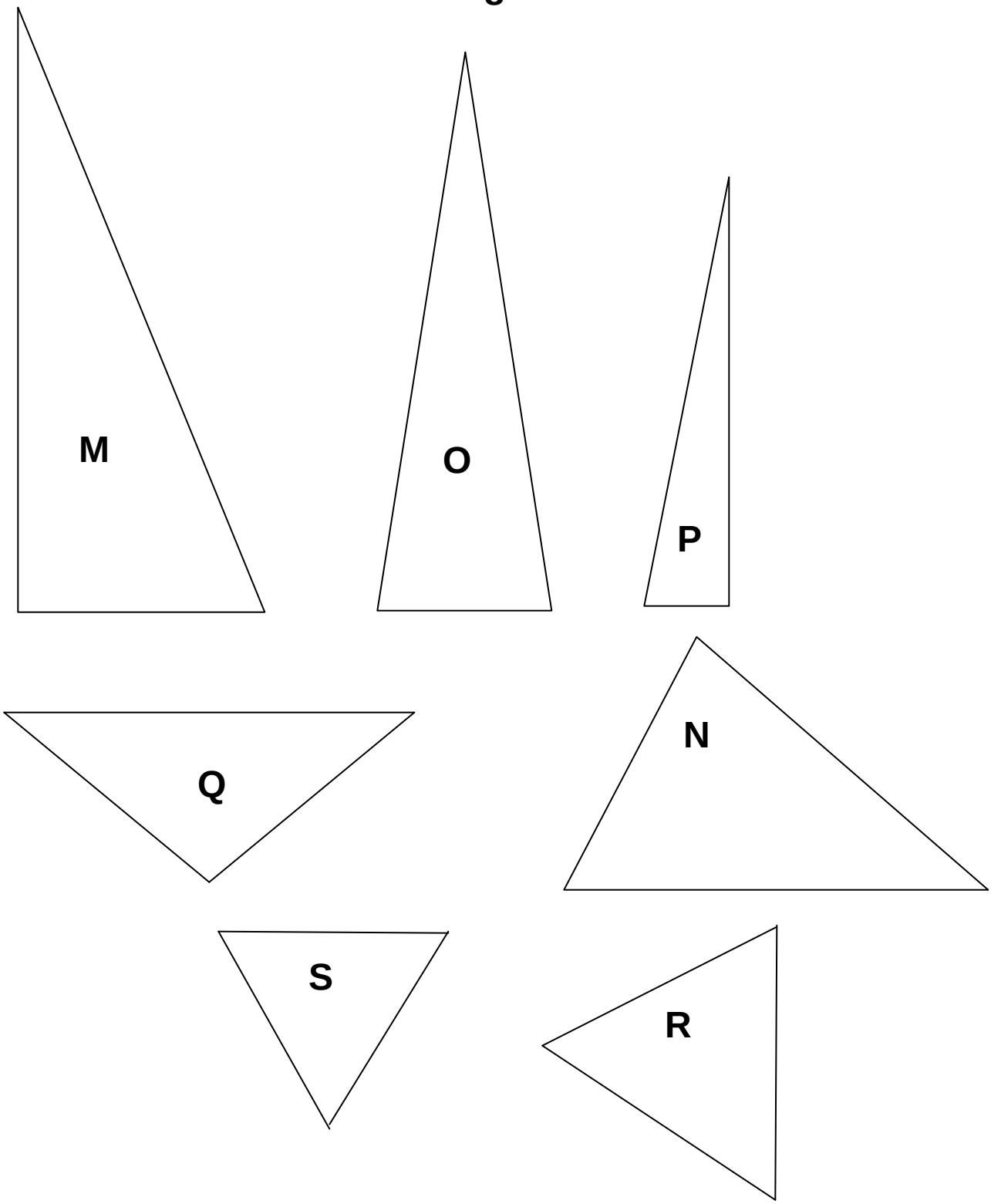
Triangle Sorting Pieces
Page 1



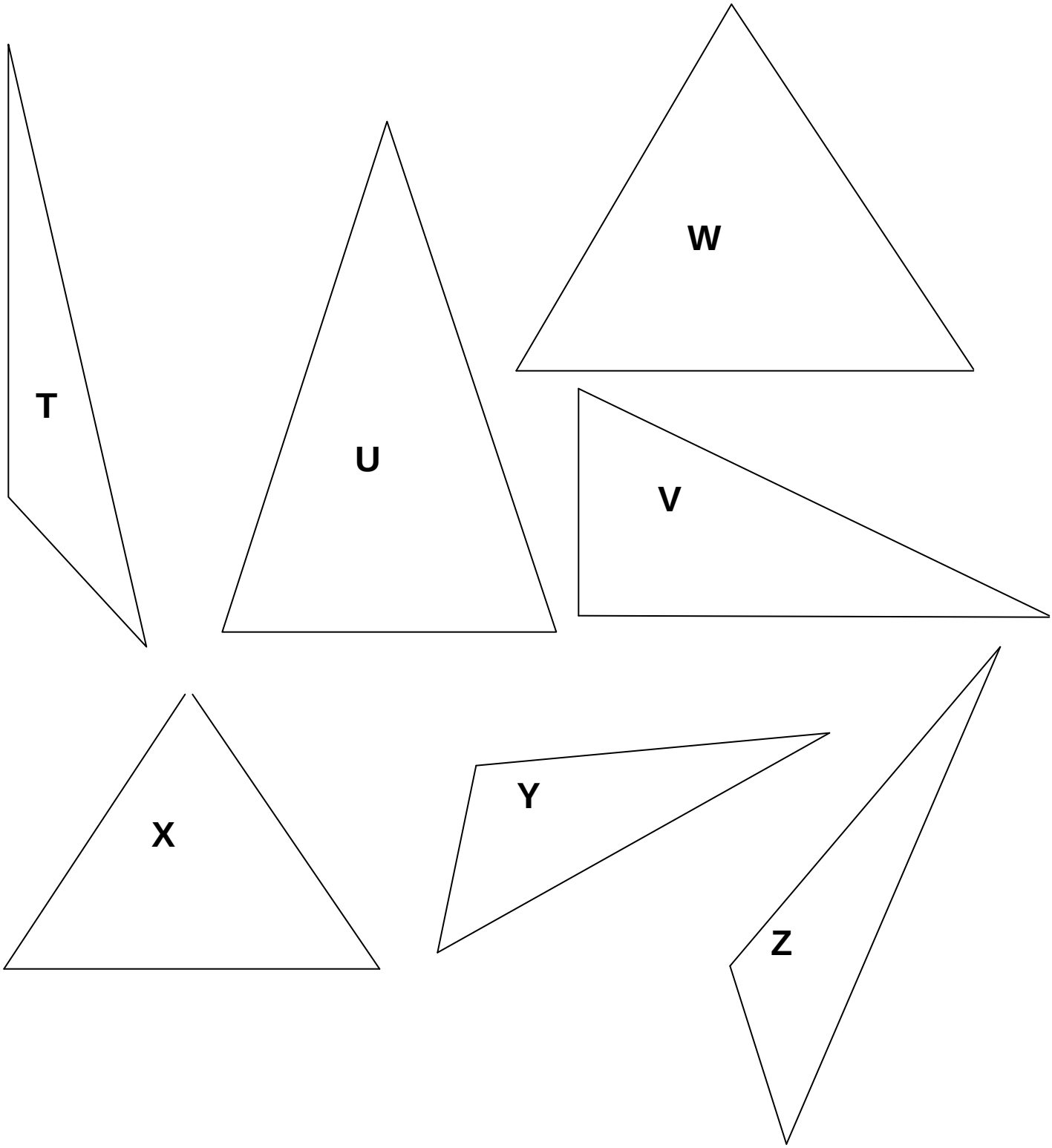
Triangle Sorting Pieces
Page 2



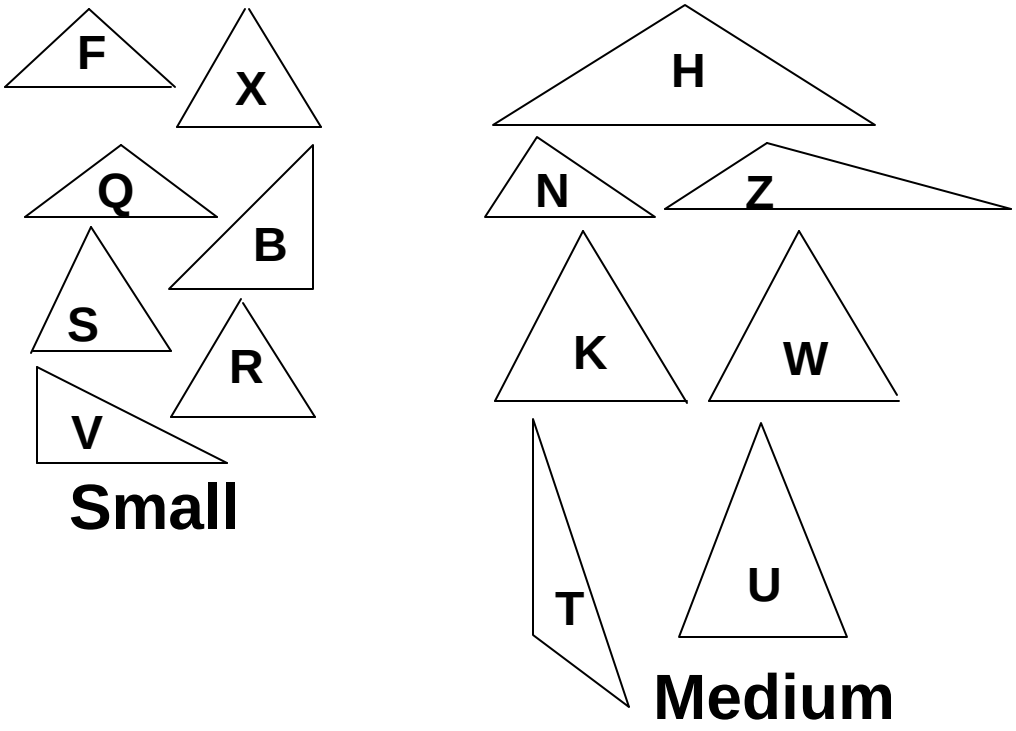
Triangle Sorting Pieces
Page 3

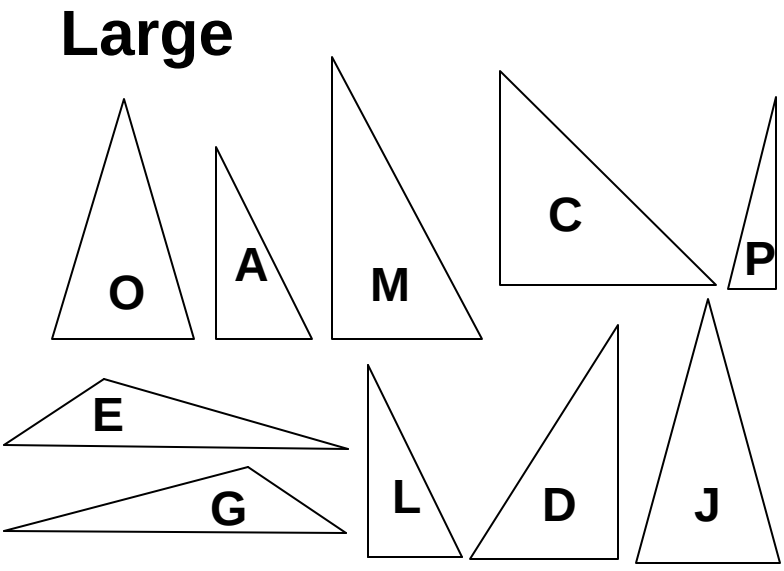


Triangle Sorting Pieces
Page 4

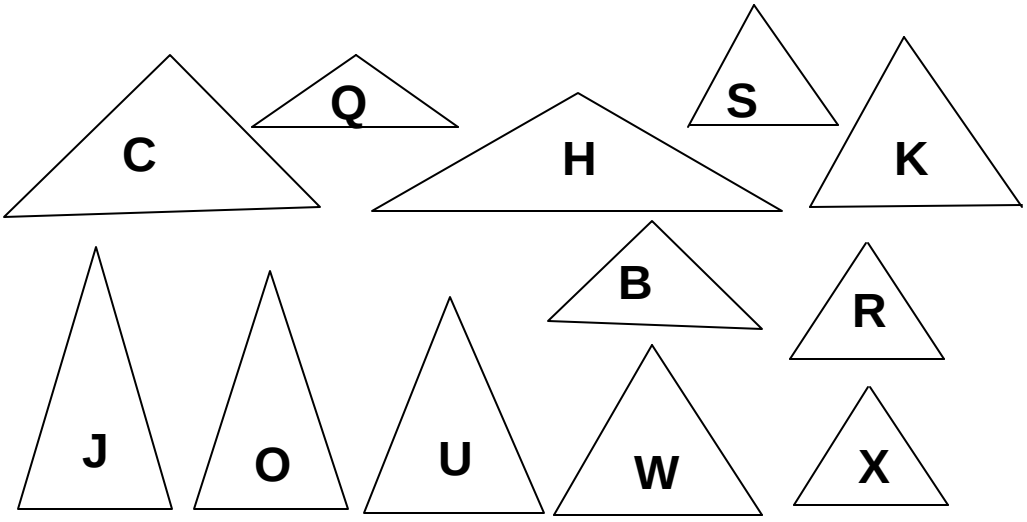


Sample Student Sort 1



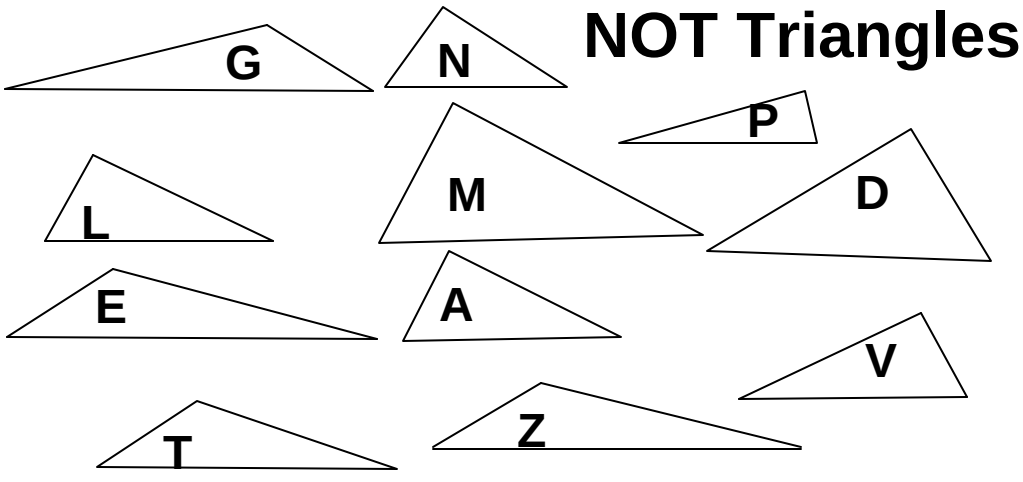


Sample Student Sort 2

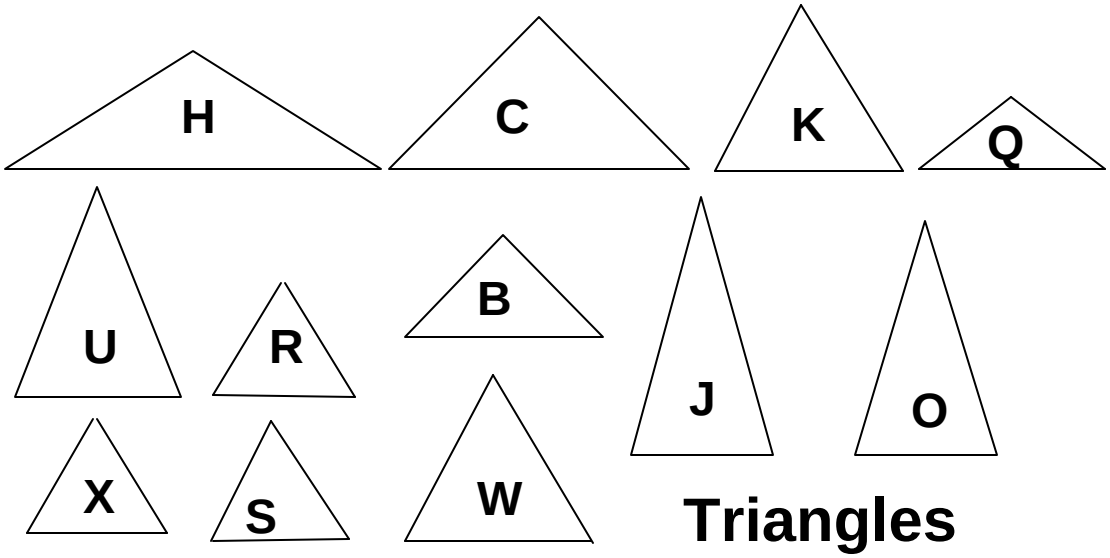


Triangles

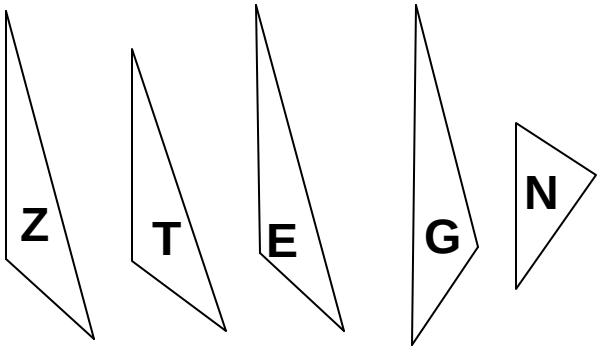
NOT Triangles



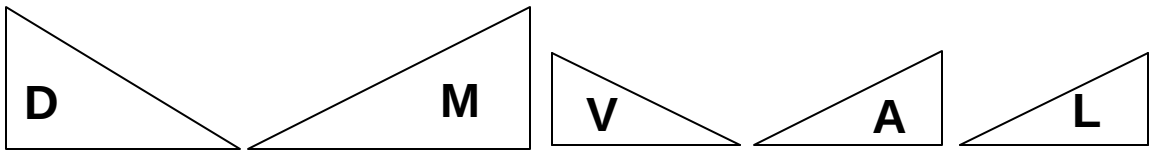
Sample Student Sort 3



Triangles

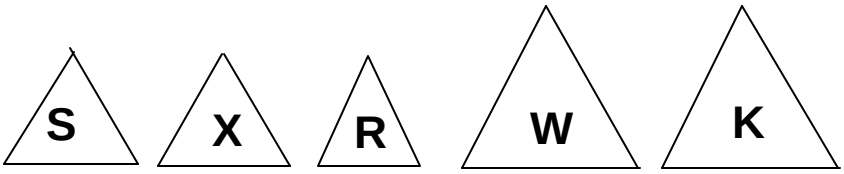


Look alike...
N is smaller...
NOT triangles

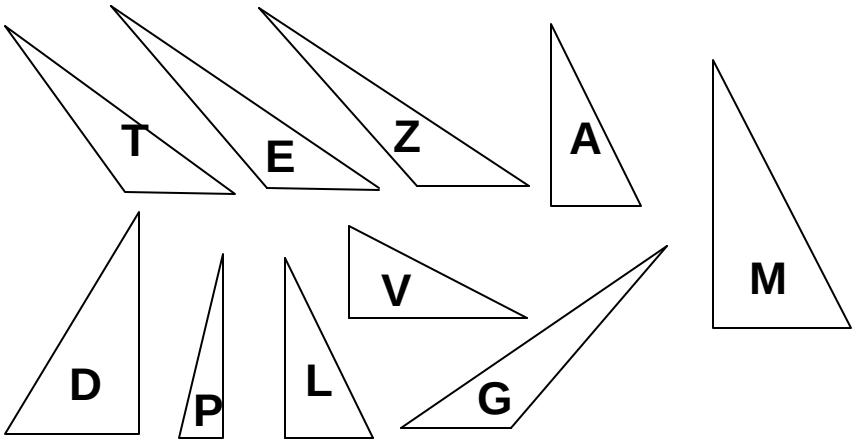


Look like ramps... NOT triangles

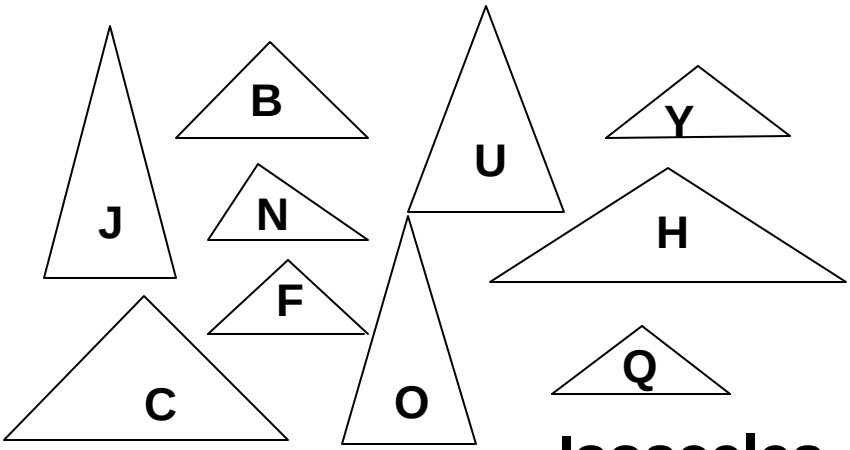
Sample Student Sort 4



Equilateral

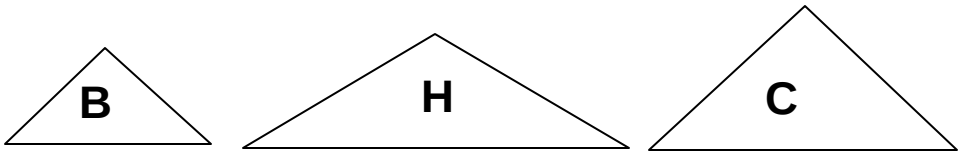


Scalene

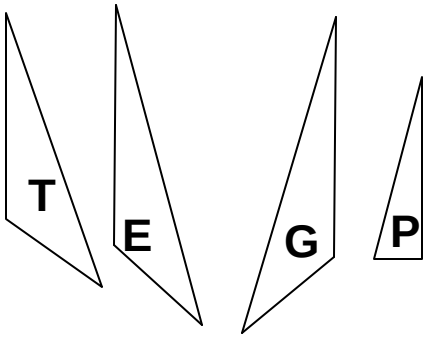


Isosceles

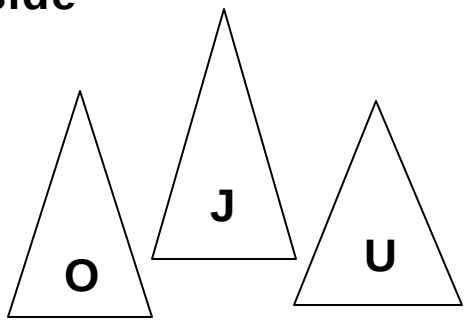
Sample Student Sort 5



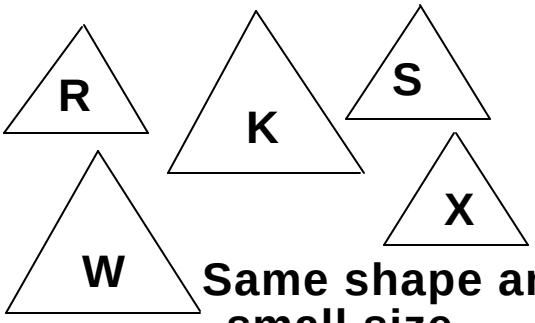
One longest side



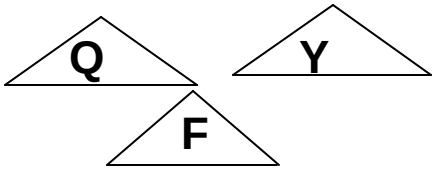
Irregular and very narrow



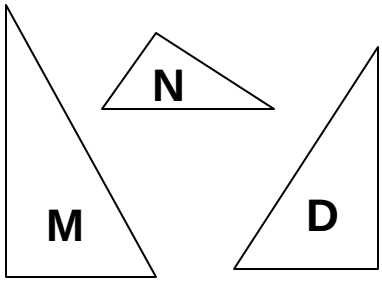
Two sides are similar, one is shorter



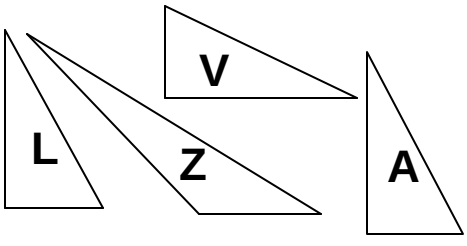
Same shape and small size



Two sides are similar, one is longer

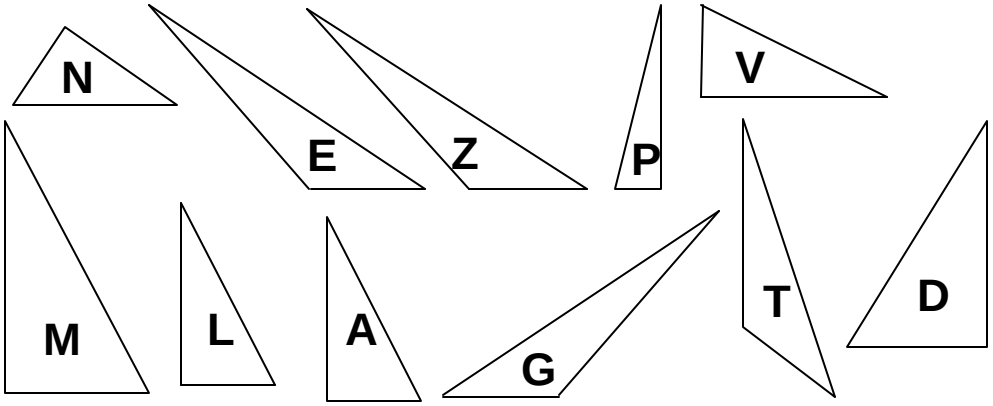


Irregular sides

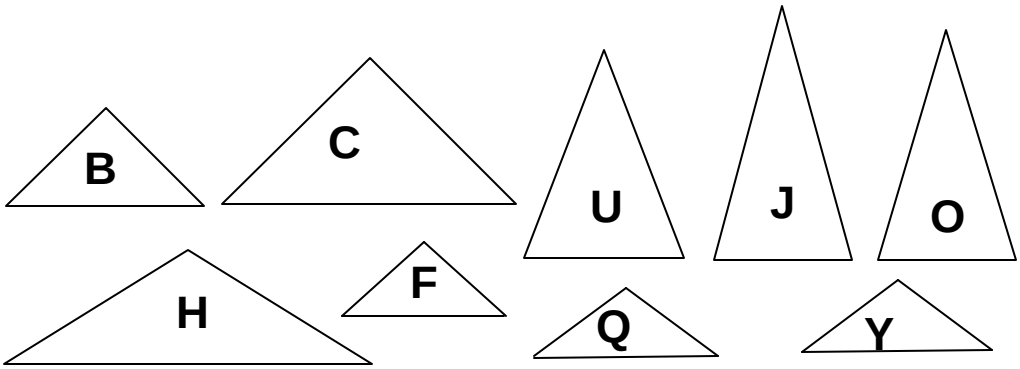


Three uneven sides

Sample Student Sort 6

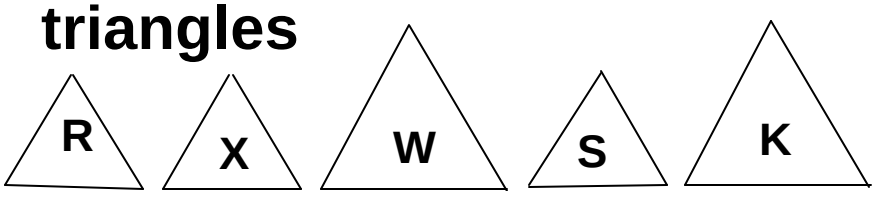


Every side has a different size



Two sides equal, third side smaller or larger

"Perfect" triangles



Topic: Quadrilaterals and their Properties

Description: Participants will explore quadrilaterals and their properties through the use of various manipulatives such as sorting pieces and geo-strips. The sequence of activities is designed to facilitate an increase in a learner's van Hiele level of thinking about quadrilaterals from Level 1 to Level 3. First, the participants learn how to determine the van Hiele levels of their own students by analyzing how they sort a set of quadrilateral pieces. Then they play the game "What's My Rule?" to develop the ability to classify quadrilaterals by various attributes and to focus on more than one attribute at a time. The participants also construct parallelograms, rectangles, rhombi, and squares using D-stix, geo-strips, toothpicks, or other manipulatives and make observations while the figures are flexed (Level 2). Finally, the participants identify relationships between parallelograms, rectangles, rhombi, squares, trapezoids, kites, and darts through a lab that culminates in the creation of a quadrilateral family tree (Level 3).
Although these activities are presented with quadrilaterals, most of them are easily adapted to triangles and other polygons.

Related SOL: K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15

Activity: Quadrilateral Sort

Format: Small Group/Large Group

Objectives: After performing their own sorts, participants will be able to distinguish the way students at various van Hiele levels of geometric thought may sort quadrilaterals.

Related SOL: K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15

Materials: Quadrilateral Sorting Pieces Activity Sheet with quadrilaterals cut out and placed in a plastic baggy or manila envelope. You will need at least one set of quadrilaterals for every three participants.

Time Required: Approximately 20 minutes

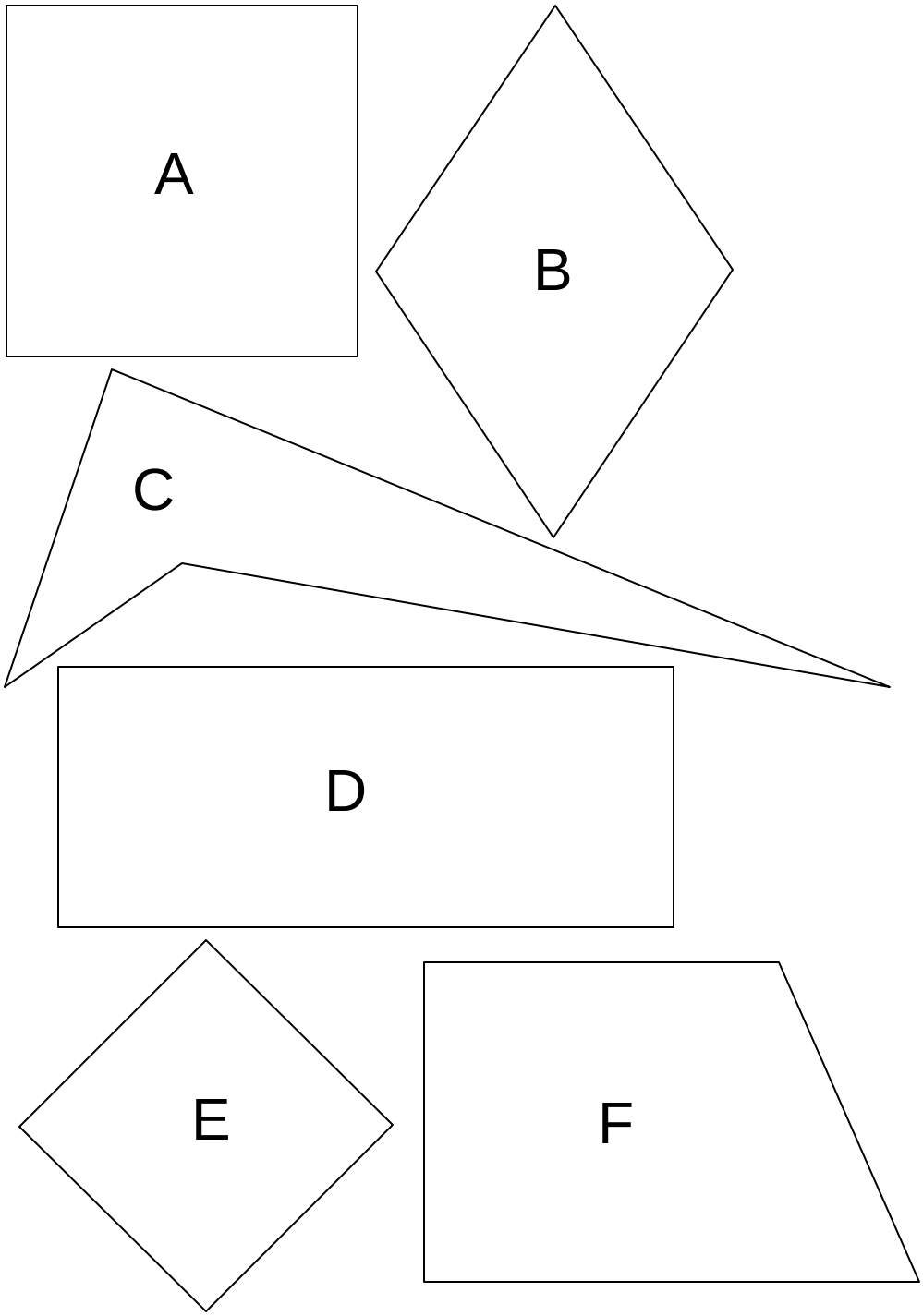
Directions:

1) Divide the participants into small groups. Distribute the sets of cut-out quadrilaterals, at least one set per three participants. Instruct the participants to lay out the pieces with the letters up. Do not call them quadrilaterals. Tell the participants that the objects can be grouped together in many different ways. For example, if we sorted the figures that make up the American flag (the red stripes, the white stripes, the blue field, the white stars), we might sort by color and put the white stripes and the stars together because they are white, the red stripes in another group because they are red, and the blue field by itself because it is the only blue object. Another way the flag parts could be grouped would be all the stripes and the blue field together because they are all rectangles and all the stars together because they are not rectangles. Have them sort the figures into groups that belong together, recording the letters of the pieces they put together and the criteria they used to sort. Have them sort two or three times, recording each sort.

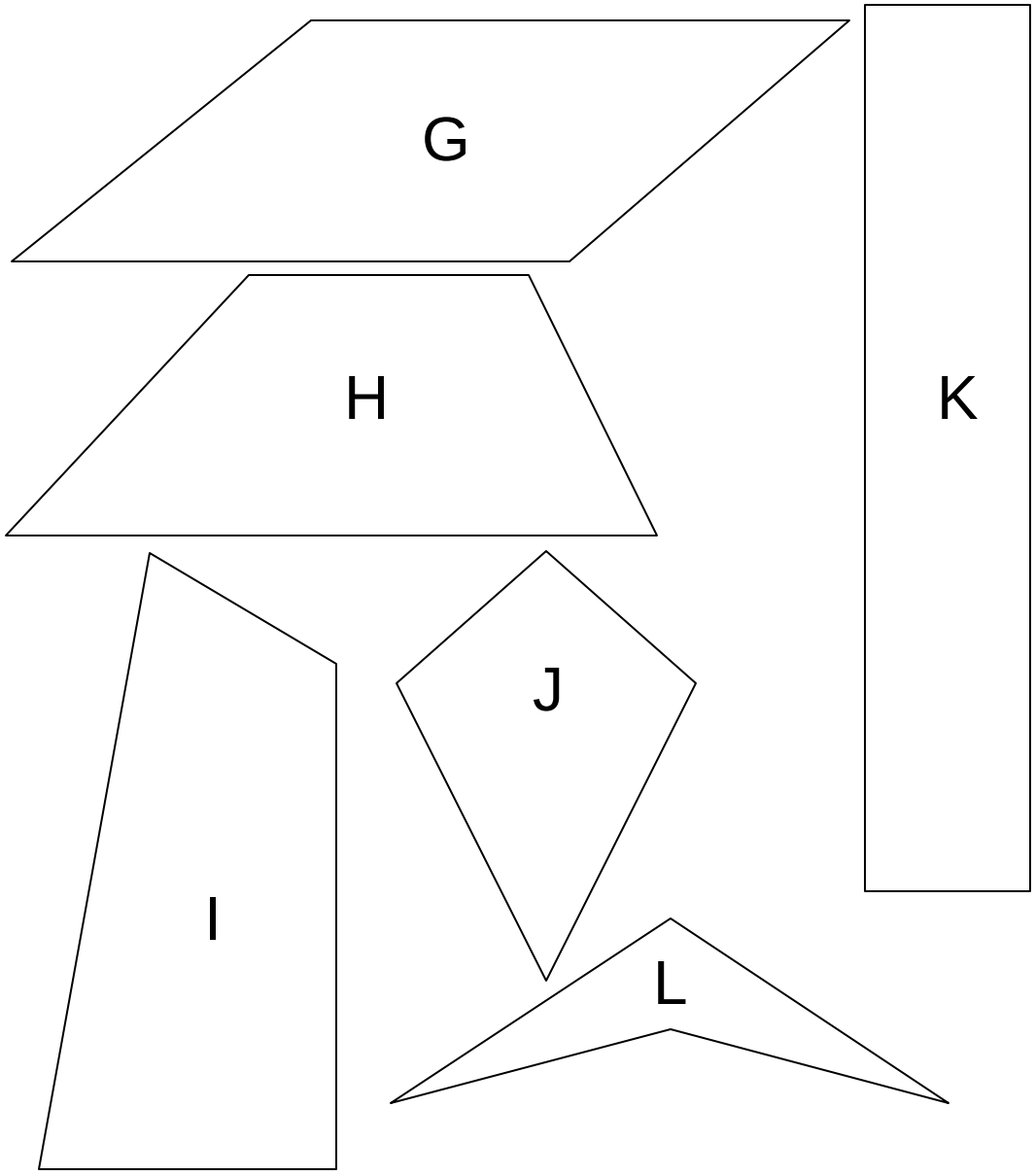
2) Ask the participants to describe their sorts. Have them compare their sorts with those of other groups.

3) Ask them how they think their students would sort these figures.

Quadrilateral Sorting Pieces



Quadrilateral Sorting Pieces
page 2



Activity: What's My Rule?

Format: Small Group/Large Group

Objectives: After playing the game, participants will classify quadrilaterals by various attributes. In children, this game develops the ability to attend to more than one characteristic of a figure at the same time.

Related SOL: K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15

Materials: Quadrilateral Sorting Pieces Activity Sheet with quadrilaterals cut out and placed in a plastic baggy or manila envelope. (Use quadrilaterals from Quadrilateral Sort Activity). You will need at least one set of quadrilaterals for every three or four participants. "What's My Rule?" Activity Sheet

Time Required: Approximately 10 minutes

Directions:

1) Divide the participants into small groups. Distribute the sets of cut-out quadrilaterals, one set per group.

2) Display "What's My Rule?" Activity Sheet and review the rules of the game. One participant in each group is the sorter. The sorter writes down a "secret rule" to classify the set of quadrilaterals into two or more piles and uses that rule to slowly sort the pieces as the other players observe.

3) At any time, the players can call "stop" and guess the rule. After the correct rule identification, the player who figured out the rule becomes the sorter. The correct identification is worth five points. A correct answer, but not the written one, is worth one point. Each incorrect guess results in a two-point penalty. The winner is the first one to accumulate ten points.

WHAT'S MY RULE?
Rules

1. Choose one player to be the sorter. The sorter writes down a "secret rule" to classify the set of quadrilaterals into two or more piles and uses that rule to slowly sort the pieces as the other players observe.

2. At any time, the players can call "stop" and guess the rule. The correct identification is worth five points. A correct answer, but not the written one, is worth one point. Each incorrect guess results in a two-point penalty.

3. After the correct rule identification, the player who figured out the rule becomes the sorter.

4. The winner is the first one to accumulate ten points.

Activity: Quadrilateral Properties Laboratory

Format: Small Group/Large Group

Objectives: Participants will construct parallelograms, rectangles, rhombi, and squares, using D-stix, geo-strips, or toothpicks and marshmallows. Participants will identify the properties of the constructed figures.

Related SOL: K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15

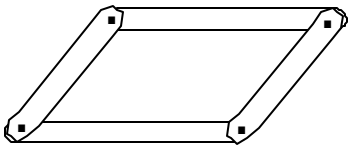
Materials: One of the following per participant: D-stix, geo-strips, or toothpicks cut into two different lengths and marshmallows; square corner (the corner of an index card or book); Types of Quadrilaterals Activity Sheet

Time Required: Approximately 20 minutes

Directions:

1) Divide the students into small groups and direct each group to experiment as you ask questions. Be sure to model constructing the quadrilaterals and flexing them.

2) Have the participants pick two pairs of congruent segments and connect them as shown below. Have them flex the figure to different positions.



Ask:

- What stays the same? (Lengths of the sides, the opposite sides are parallel, opposite angles are congruent, sum of the measures of the angles, perimeter)
- What changes? (Size of angles, area, lengths of diagonals)
- What do you notice about the opposite sides of this quadrilateral? (They remain parallel and congruent.)

_____

A **parallelogram** is a quadrilateral with both pairs of sides parallel.

_____

- What is the sum of the measures of the interior angles of this quadrilateral? (360°)

- What do you notice about the opposite angles? (Congruent)

Note to Trainer: Some participant will likely turn the strips so that they cross, forming two triangles. If no one does, you should. Ask if this figure is a polygon. Elicit from the group what the essential elements of a polygon are, i.e.,

a) composed of line segments
b) simple (the segments don't cross)
c) closed
d) lies in a plane (e.g., if you take a wire square and twist it so that it isn't flat, it is no longer a polygon)

3) Make one of the angles a right angle (You can use the square corner to check your accuracy.)
Ask:
   - What happens to the other angles? (They become right angles.)
   - Will this always be true when you make one angle of a parallelogram a right angle? (Yes)
   - How do you know? (The sum of the measure of the angles in a parallelogram is 360°. One angle measures 90°. Its opposite angle must measure the same or 90°. Subtracting these two angles from 360°, the remaining two angles, which are congruent since they are opposite angles in a parallelogram, must have a total measure of 180°. Therefore, each angle measure is 90°. Note: This is Level 3 thinking.)
   - Is it still a parallelogram? (Yes)
   - Is it still a quadrilateral? (Yes)
   - Is it still a polygon? (Yes)
   - What other name, besides polygon, quadrilateral, and parallelogram, can be given to it now? (rectangle)

   _____

   A **rectangle** is a parallelogram with four right angles.

   _____

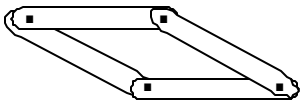
4) Make a parallelogram that has all four sides equal in length. What is another name for this parallelogram? (rhombus)

   _____

   A **rhombus** is a parallelogram with four congruent sides.

   _____

5) Flex the figure to different positions.



Ask:

- What stays the same? (Lengths of the sides, the opposite sides are parallel, opposite angles are congruent, sum of angles, perimeter)
- What changes? (Size of angles, area, lengths of diagonals)
- What is the sum of the measures of the interior angles of this quadrilateral? (360°)
- What do you notice about the opposite angles? (Congruent)
- Is it still a quadrilateral? (Yes)
- Is it still a polygon? (Yes)

6) Make one of the angles of this rhombus a right angle, checking with your square corner.
Ask:

- What happens to the other angles? (All right angles)
- Is it still a parallelogram? (Yes)
- What other name, besides polygon, quadrilateral, parallelogram, and rhombus, can be given to this new figure? (square)

_____

A **square** is a parallelogram with four congruent sides and four right angles.

_____

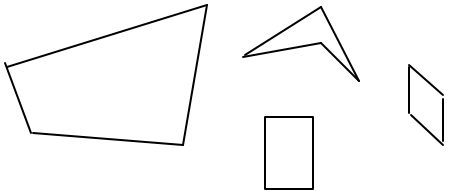
Ask:

- Is it a rectangle? (Yes)
- How do you know? (It has four right angles.)

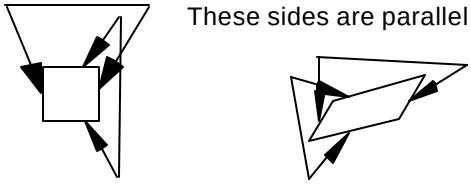
7) Distribute Types of Quadrilaterals Activity Sheet and discuss the definitions for quadrilateral, parallelogram, rectangle, rhombus, and square. Discuss the examples of each, noticing their orientations and how each example fits the definition even though they are not necessarily the stereotypical figure usually seen. Discuss the implications for teaching a Level 1 student who recognizes figures by comparing them to a known figure. This type of student might describe a rectangle by saying, “I know it’s a rectangle because it looks like a door.”

TYPES OF QUADRILATERALS

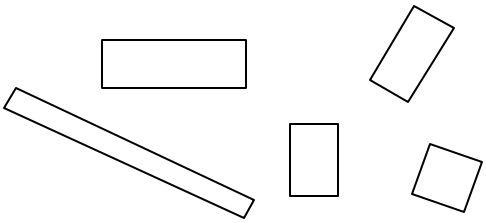
A **quadrilateral** is a **four**-sided polygon.



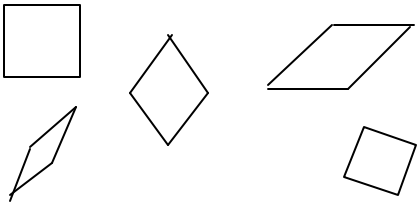
A **parallelogram** is a quadrilateral with both pairs of **opposite** sides **parallel**.



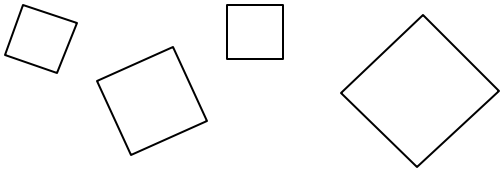
A **rectangle** is a quadrilateral with **four right angles**.



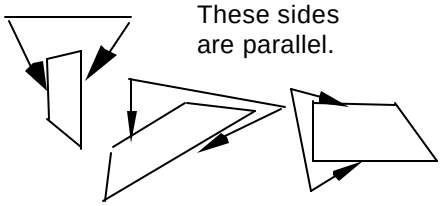
A **rhombus** is a quadrilateral with **four sides congruent**.



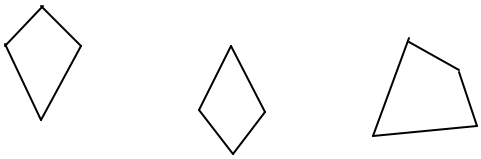
A **square** is a quadrilateral with **four right angles** and **four congruent sides.**



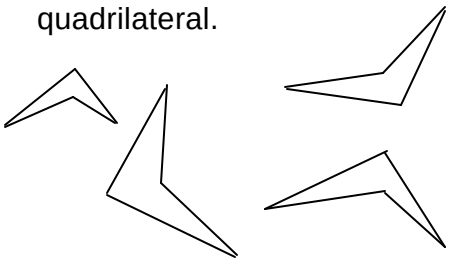
A **trapezoid** is a quadrilateral with **exactly one** pair of parallel sides.



A **kite** is a convex quadrilateral with two distinct pairs of **adjacent congruent sides**.



A **dart** is a **concave** quadrilateral.



Activity: Quadrilateral Sorting Laboratory

Format: Small Group/Large Group

Objectives: Participants will record which quadrilaterals meet the various descriptions listed in the properties table, determine which sets are identical and are subsets of one another, attach labels to each category, and create a quadrilateral family tree.

Related SOL: K.11, K.12, K.13, 1.16, 2.22, 3.18, 4.15, 4.17, 5.14, 5.15

Materials: Quadrilateral Sorting Pieces Activity Sheet with quadrilaterals cut out and placed in a plastic baggy or manila envelope. (Use quadrilaterals from Quadrilateral Sort Activity). Quadrilateral Sorting Laboratory Activity Sheet, Quadrilateral Table Activity Sheet, Quadrilateral Family Tree Activity Sheet

Time Required: Approximately 30 minutes

Directions:

1) Distribute Quadrilateral Sorting Laboratory Activity Sheet, Quadrilateral Table Activity Sheet and the Quadrilateral Family Tree Activity Sheet. Divide the participants into small groups and direct each group to experiment and answer the questions, using their quadrilateral sorting pieces.

2) After the participants have filled out the Quadrilateral Table Activity Sheet, have pairs of groups compare their answers, and reconcile any discrepancies.

3) Have the participants continue with Steps 5-10. Refer to the Quadrilateral Table as needed while discussing the results of #10.

4) For Step 11 the participants can construct the family tree as small groups or as a large group. Discuss various possibilities for the entries.

Quadrilateral Sorting Laboratory

Directions:

1) Spread out your quadrilateral pieces with the letters facing up so you can see them.

2) Find all of the quadrilaterals having four right angles. List them by letter alphabetically in the corresponding row of the Quadrilateral Table.

3) Consider all of the quadrilaterals again. Find all of the quadrilaterals having exactly one pair of parallel sides. List them by letter alphabetically in the corresponding row of the Quadrilateral Table.

4) Continue in this manner until the Quadrilateral Table is complete.

5) Which category is the largest? What name can be used to describe this category?

6) Which lists are the same? What name can be used to describe quadrilaterals with these properties?

7) Are there any lists that are proper subsets of another list? If so, which ones?

8) Are there any lists that are not subsets of one another that have some but not all members in common? If so, which ones?

9) Which lists have no members in common?

10) Label each of the categories in the Quadrilateral Table with the most specific name possible using the labels kite, quadrilateral, parallelogram, rectangle, rhombus, square, and trapezoid. For example, #1 - a quadrilateral that has four right angles is a rectangle. (Having four right angles is not enough to make it a square; it would need four congruent sides as well.)

11) Compare your results to that of the other groups. Then fill out the family tree by inserting the names kites, rectangles, squares, and trapezoids into the appropriate places on the diagram.

Quadrilateral Table

1. has four right angles

2. has exactly one pair of parallel sides

3. has two pairs of opposite sides congruent

4. has four congruent sides

5. has two pairs of opposite sides parallel

6. has no sides congruent

7. has two pairs of adjacent sides congruent, but not all sides congruent

8. has perpendicular diagonals

9. has opposite angles congruent

10. is concave

11. is convex

12. its diagonals bisect one another

13. has four sides

14. has four congruent angles

15. has four congruent sides and four congruent angles

Quadrilateral Family Tree